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PULL-IN PERFORMANCE OF FIRST-ORDER PHASE-LOCKED LOOPS

TECHNICAL DOCUMENTARY REPORT NO. ESD-TDR-64-640

FEBRUARY 1965

P. Bratt

Prepared for

DIRECTORATE OF AEROSPACE INSTRUMENTATION
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

Project 705. 2
Prepared by

THE MITRE CORPORATION
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FOREWORD

The writer would like to express his appreciation to the late Dr. F. Moskowitz of MITRE's Department D-80 for his continuing advice and interest during the preparation of this paper.

ABSTRACT

A phase-locked loop is briefly described, together with the derivation of the basic differential equation which governs the dynamic behavior of the loop during the pull-in process. The special case of the pull-in process of the first-order loop when a sine wave of constant frequency is applied to the input of the loop is also described. The relationship between the frequency mistuning of the loop, the initial starting phase angle of the input sine wave, and the time required for the loop to pull in is discussed. The statistical parameters associated with the pull-in time is reviewed. In particular, expressions are given for the probability density function of the pull-in time and the cumulative distribution of the pull-in time.

REVIEW AND APPROVAL

Publication of this technical documentary report does not constitute Air Force approval of the reports findings or conclusions. It is published only for the exchange and stimulation of ideas.

Roy D. Ragsdale

ROY D. RAGSDALE
Colonel, USAF
Director, Aerospace Instrumentation

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SECTION I

INTRODUCTION

The ability of phase-locked loops to separate narrow-band signals from wide-band noise has led to their increasing use as FM discriminators, tracking filters, and synchronizing devices, particularly in space communications.

It is possible to design a phase-locked loop so that the loop need have an effective bandwidth only large enough to pass the difference between the input signal frequency and the estimate of this frequency as generated by an oscillator within the loop. Since this difference frequency has considerably less variation than the actual input signal, the loop does not need nearly as large a bandwidth as would be needed if the loop were merely a tuned circuit placed between the system input and the output, which would have to pass all frequencies over which the input signal varied.

Since the bandwidth of the phase-locked loop can be made much smaller than that of a comparable nontracking filter, the amount of noise reaching the output is considerably smaller, and the loop develops a greater resistance to noise at the input.

However, improvement in one aspect of a system usually results in a sacrifice in another, and phase-locked loops are no exception. The reduction in the effective noise bandwidth results in an increase in the pull-in time of the loop, that is an increase in the time between the appearance of a signal at the input and satisfactory operation of the loop.

The pull-in time can usually be neglected for systems having long periods of continuous communication time available, as would the case for a stationary ground-based system with direct line of sight. However, instances do arise where the pull-in time of the loop may be a significant fraction of the communication time available. Two examples that come to mind are a meteor-burst scatter system, where the ionized trails only last for a few seconds, and a military satellite communication system subject to pulse jamming, where a straightforward defense against the jammer is achieved by gating the satellite receiver on and off in a pseudo-random fashion.

With these comments in mind, it is apparent that the pull-in time of a phase-locked loop may be a significant factor in the design of certain communication systems. To illustrate the problem of pull-in time, the example of the simplest type of phase-locked loop is chosen, and the time required for the loop to pull-in when an input signal of constant frequency is applied to the input is derived.

SECTION II

DESCRIPTION OF PHASE-LOCKED LOOP

The basic elements of a typical phase-locked loop are shown in Fig. 1 and consist of: a voltage controlled oscillator (VCO) with a nominal frequency equal to that of the input to the loop; a phase detector which compares the phase of the output of the oscillator with that of the input to the loop; an amplifier which amplifies the output voltage of the phase detector; and a low-pass filter which filters the output voltage of the amplifier before it is applied to the VCO.

Briefly, the phase-locked loop operates as follows: The phase detector beats the signal input and the VCO output together, giving a low-frequency output proportional to the sine of the phase difference between the two signals, together with a high-frequency component located at the sum frequency of the two inputs. The low-pass filter accepts only the low-frequency term, which is applied as a control voltage to the VCO, forcing the output phase of the VCO to follow the input signal phase. Figures 2 and 3 show the control characteristics of the phase detector and the voltage-controlled oscillator. The phase detector generates a voltage $e_1(t)$ which is proportional to both the amplitudes of the two inputs signals and to the sine of their phase difference, the constant of proportionality being K_θ. (K_θ has dimensions of $(\text{volts} \times \text{radians})^{-1}$.) The voltage-controlled oscillator shifts its output frequency by an amount proportional to the voltage applied to its control input, the constant of proportionality being K_{vco} (radians/sec/volt). The point about which the frequency of the

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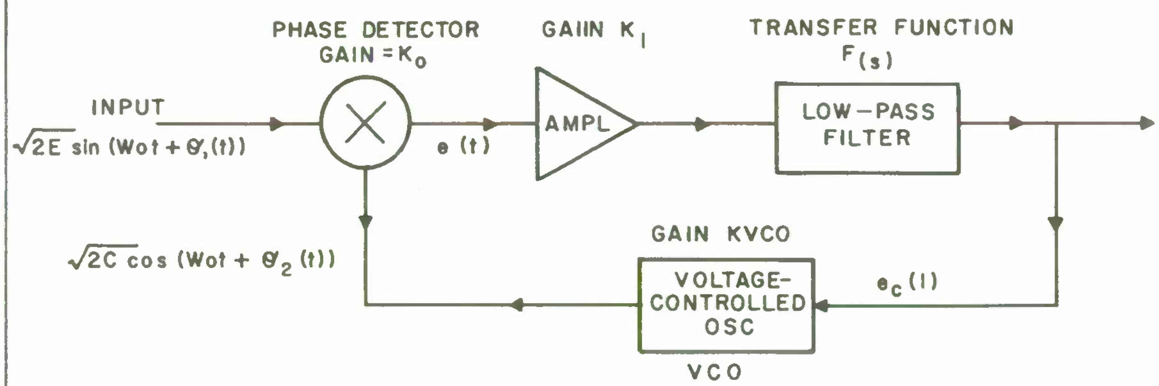

Fig. 1. Block Diagram of a Phase-Locked Loop

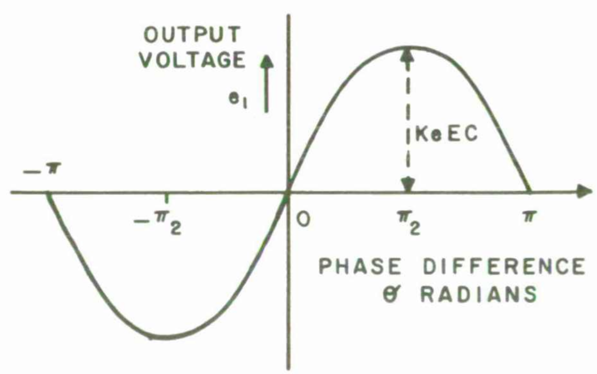

Fig. 2. Control Characteristic of the Phase Detector

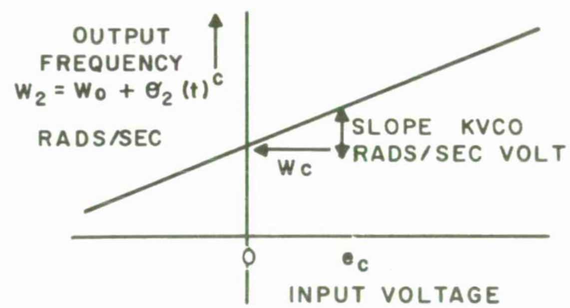

Fig. 3. Control Characteristic of the Voltage-Controlled Oscillator

VCO shifts is ω_o, usually referred to as the natural or free-running frequency of the VCO. Figure 3 indicates that ω_o is the output frequency of the VCO when the control voltage is zero.

The differential equation governing the dynamic behavior of the loop is derived below. The input signal to the loop is assumed to be $\sqrt{2}\ E \sin(\omega_o + \theta_1(t))$, where E is the rms amplitude and $\theta_1(t)$ represents the variation of its phase as a function of time. The output of the voltage-controlled oscillator is assumed to be $\sqrt{2}\ C \cos(\omega_o + \theta_2(t))$, where C is the rms amplitude and represents the variation of its phase as a function of time.

A natural question at this point is why are the loop-input frequency and the VCO frequency expressed in terms of ω_o plus a time-varying phase angle rather than in terms of two different frequencies, for instance, ω_1 and ω_2. When the loop input is frequency modulated, both the input and VCO frequencies change as a function of time during and after the condition of lock-on. If the input frequency is held constant, the VCO frequency changes as a function of time during the lock-on process, and then remains constant. Therefore, both frequencies generally vary as a function of time. Hence, it is very convenient (in a mathematical sense) to express both frequencies in terms of a frequency which is fundamental to the system and which always remains constant. The natural frequency of the VCO, ω_o, is just such a frequency. For example, the input to the loop was written as

$$\sqrt{2}\ E \sin[\omega_o t + \theta_1(t)] \tag{1}$$

Suppose the loop input is a sinusoid of constant frequency ω_1 given by

$$\sqrt{2}\ E \sin(\omega_1 t + a)$$

We could just as correctly write it as

$$\sqrt{2}\ E \sin [\omega_o t + (\omega_1 - \omega_o)\, t + a] \tag{2}$$

Which when compared with Eq. (1) gives

$$\theta_1 (t) = (\omega_1 - \omega_o)\, t + a$$

Therefore, there is nothing lost by expressing signals of different frequencies all in terms of the same frequency, provided the necessary adjustment is made in the time-varying phase angle.

Continuing the development of the differential equation, the output of the phase detector is given by

$$\begin{aligned} e_1 (t) &= K_\theta \sqrt{2}\ E \sin [\omega_o t + \theta_1(t)]\ \sqrt{2}\ C \cos [\omega_o t + \theta_2(t)] \\ &= K_\theta\ EC \{ \sin [2\omega_o t + \theta_1(t) + \theta_2(t)] + \sin [\theta_1(t) - \theta_2(t)] \} \end{aligned} \tag{3}$$

The term $2\,\omega_o$ in Eq. (3) can be neglected, since neither the low-pass filter nor the VCO will respond to it, provided ω_o is reasonable large.

The output of the filter is then given by

$$e_c(t) = K_1\ F(S)\, K_\theta\, EC \sin [\theta_1(t) - \theta_2(t)] \tag{4}$$

where $F(S)$ represents, in operational notation, the effect of the filter on the signal $\sin [\theta_1 (t) - \theta_2 (t)]$.

Let

$$\theta_1(t) - \theta_2(t) = \theta(t) \qquad (5)$$

where $\theta(t)$ is the instantaneous phase difference between the input signal and the output of the VCO.

Equation (4) then becomes

$$e_c(t) = K_1 \, F(S) \, K_\theta \, EC \sin\theta(t) \qquad (6)$$

The output frequency of the VCO is given by

$$\omega_o + \dot{\theta}_2(t) = \omega_o + K_{vco} \, e_c(t) \qquad (7)$$

Substituting for $e_c(t)$ from Eq. (6) gives

$$\dot{\theta}_2(t) = K_{vco} \, K_1 \, K_\theta \, EC \, F(S) \sin\theta(t) \qquad (8)$$

Differentiating Eq. (5) with respect to time gives

$$\dot{\theta}_2(t) = \dot{\theta}_1(t) - \dot{\theta}(t) \qquad (9)$$

Substituting in Eq. (7) for $\dot{\theta}_2(t)$ gives

$$\dot{\theta}_1(t) - \dot{\theta}(t) = K_{vco} \, K_1 \, K_\theta \, EC \, F(S)$$

or

$$\dot{\theta}(t) = \dot{\theta}_1(t) - K_{vco}\, K_1\, K_\theta\, EC\, F(S) \sin\theta(t) \tag{10}$$

Let

$$K = K_{vco}\, K_\theta\, K_1\, EC$$

Equation (10) may now be rewritten as

$$\dot{\theta}(t) = \dot{\theta}_1(t) - KF(S) \sin\theta(t) \tag{11}$$

Equation (11) is the basic differential governing the behavior of the phase-locked loop where

$\dot{\theta}(t)$ = instantaneous frequency difference between the input to the loop and the output of the VCO.

$\dot{\theta}_1(t)$ = instantaneous frequency difference between the input to the loop and the free-running frequency of the VCO.

$\theta(t)$ = instantaneous phase difference between the input to the loop and the output of the VCO.

The terms "instantaneous phase difference" and "instantaneous frequency difference" are discussed in Appendix V.

SECTION III

CONSIDERATION OF FIRST-ORDER PHASE-LOCKED LOOP

The simplest type of phase-locked loop is that shown in Fig. 1, except that the low-pass filter is omitted, a direct connection being made between the output of the amplifier and the input to the VCO $[F(S) = 1]$. Although the low-pass filter is not present in the feed-back loop, it is assumed that the VCO will not respond to high-frequency components applied to its input (that is, the term in $2\omega_o$ resulting from product detection).

We now consider the case where a sine wave of constant frequency ω_1 is applied to the loop.

For the case discussed above, Eq. (11) takes the particularly simple form given by

$$\dot{\theta}(t) = \Delta\omega - K \sin\theta(t) \tag{12}$$

where

$\theta(t)$ = instantaneous phase difference between the input to the loop and the output of the VCO.

$\dot{\theta}(t)$ = instantaneous frequency difference between the input to the loop and the output of the VCO.

$\Delta\omega = \omega_1 - \omega_o$ = frequency difference between the input to the loop and the free-running frequency of the VCO.

K = open-loop gain of the loop.

Equation (12) is a differential equation of first order, so that the phase-locked loop shown in Fig. 1 (with the low-pass filter omitted) is usually referred to as a first-order loop. Basically, Eq. (12) states that when a sine wave of constant frequency is applied to the first-order loop, the sum of the instantaneous frequency difference, plus K times the sine of the instantaneous phase difference, is always constant, that is

$$\dot{\theta}(t) + K \sin\theta(t) = \Delta\omega \tag{13}$$

In general, there will be both a frequency difference and an initial phase difference between the input to the loop and the free-running frequency of the VCO. The process of pulling the VCO into frequency alignment with the input to the loop will, therefore, require finite time. Further on in this report, the pull-in time is shown to be a function of both the initial frequency detuning $(\omega_1 - \omega_o)$ and the initial phase angle θ_i.

A plot of Eq. (12) helps in understanding the physical process by which the phase-locked loop pulls into frequency alignment with the input; it is shown plotted in Fig. 4.

The first point to note is that the system represented by Eq. (12) is in equilibrium when the instantaneous frequency $\dot{\theta}(t)$ is zero.

This occurs when

$$\Delta\omega - K \sin\theta = 0 \tag{14}$$

or

$$\sin\theta = \frac{\Delta\omega}{K}$$

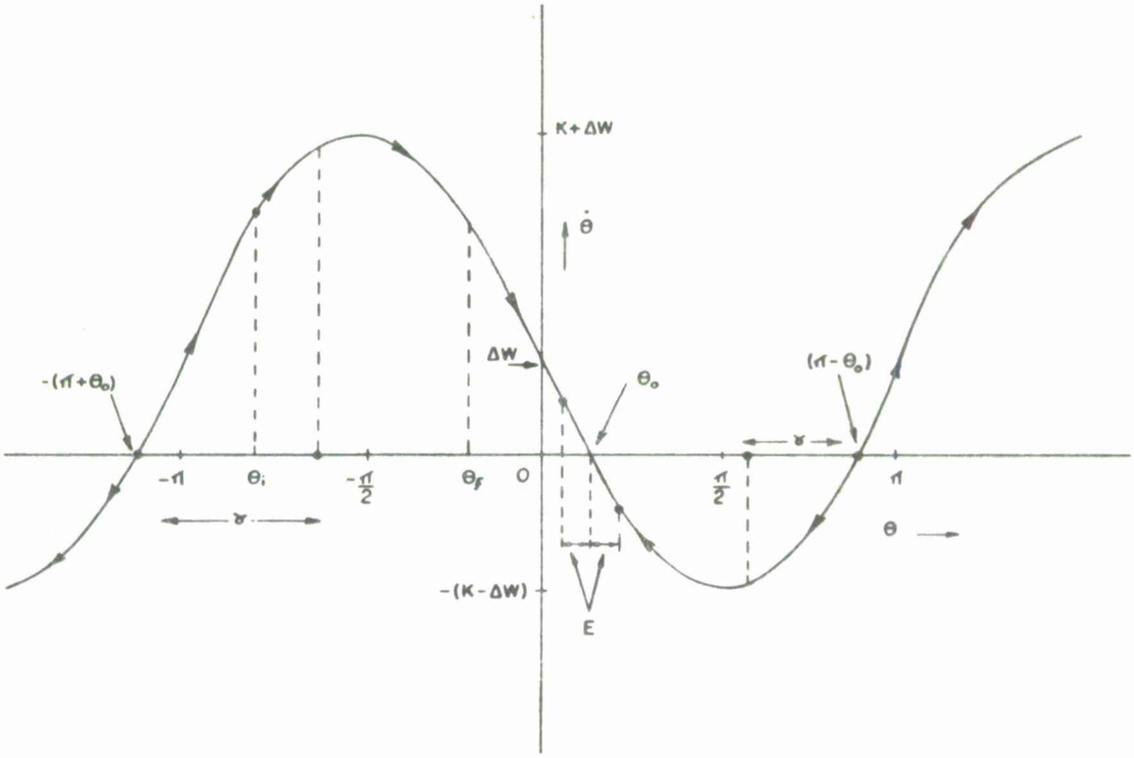

Fig. 4. Plot of $\dot{\theta}$ Versus θ for a First-Order Phase-Locked Loop

Let

$$\sin^{-1} \frac{\Delta\omega}{K} = \theta_o$$

Then the system defined in Eq. (12) is in equilibrium for values of θ given by

$$\theta = n\pi + (-1)^n \theta_o \tag{15}$$

where n is an integer including zero.

The values of θ given by Eq. (15) obviously occur at the intersections of the curve with the $\dot{\theta}(t) = 0$ axis.

Which of the values of θ given by Eq. (15) represent positions of stable equilibrium? For $\dot{\theta}$ positive the rate of change of θ with time is positive, implying that θ is increasing. On the other hand, $\dot{\theta}$ negative implies that θ is decreasing with time.

These two conditions indicate that for any phase position for which the curve is above the $\dot{\theta} = 0$ axis (Fig. 4), θ moves to the right (θ increasing), while for phase positions below the $\dot{\theta} = 0$ axis, θ moves indicate the directions in which θ moves with time. These conditions indicate that values of θ in Eq. (15) for which n is even are points of stable equilibrium, while those for which n is odd are points of unstable equilibrium. Since the curve drawn in Fig. 4 is periodic, it is sufficient to restrict our attention to any convenient 2π radians.

If the mistuning $\Delta\omega$ is increased, the phase trajectory shown in Fig. 4 moves upwards with respect to the $\dot{\theta} = 0$ axis until the two values of θ (stable condition and unstable condition) coincide. For this value of θ, say θ_{crit},

$$\theta_{crit} = \sin^{-1} \frac{\Delta\omega}{K} = \pi - \sin^{-1} \frac{\Delta\omega}{K}$$

or

$$\sin^{-1} \frac{\Delta\omega}{K} = \frac{\pi}{2}$$

$$\frac{\Delta\omega}{K} = 1$$

Hence, for the case where the mistuning $\Delta\omega$ equals the open-loop gain K, the system reaches a position of unstable equilibrium where any slight disturbance of the phase due to noise within the loop may cause the phase to recycle to the next point of unstable equilibrium.

The above considerations lead to the conclusion that the mistuning $\Delta\omega = (\omega_1 - \omega_o)$ should be less than K rads/sec so that the first-order phase-locked loop settles down to a steady condition; i. e.,

$$|\Delta\omega| = |\omega_1 - \omega_o| < K \text{ radians/sec} \tag{16}$$

For the condition given by Eq. (16), the steady-state phase error θ_o is given by

$$\theta_o = \sin^{-1} \frac{\Delta\omega}{K}$$

SECTION IV

PULL-IN TIME FOR LOOP

The time required for the loop to pull-in is given by the time required for the phase difference θ to traverse the phase trajectory from some initial value θ_i to its steady-state value θ_o. The time required to reach θ_o is always infinite and independent of θ_i, since as θ approaches θ_o, θ changes very slowly (i.e., $\dot{\theta}$ approaches zero, see Fig. 4). This point is discussed fully in Appendix I.

It is more realistic to say that pull-in time is the time required for θ to change from some initial value θ_i to $(\theta_o - \epsilon)$ or $(\theta_o + \epsilon)$ depending upon θ_i lying to the left or right of θ_o (see Fig. 4). Appendix I shows that the pull-in time T is given by

$$T = \frac{\sec\theta_o}{K} \ln \left[\frac{\cos\left(\theta_o - \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}} \; \frac{\sin\left(\frac{\theta_o - \theta_i}{2}\right)}{\cos\left(\frac{\theta_o + \theta_i}{2}\right)} \right] \tag{17}$$

for $-(\pi + \theta_o) < \theta_i < \theta_o - \epsilon$; and by

$$T = \frac{\sec\theta_o}{K} \ln \left[\frac{\cos\left(\theta_o + \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}} \; \frac{\sin\left(\frac{\theta_i - \theta_o}{2}\right)}{\cos\left(\frac{\theta_i + \theta_o}{2}\right)} \right] \tag{18}$$

for $(\theta_o + \epsilon) < \theta_i < (\pi - \theta_o)$.

If the initial value of θ_i lies in the interval $(\theta_o - \epsilon) < \theta_i < \theta_o + \epsilon$, the pull-in time is, of course, zero, since the phase angle θ_i is already within the region defining the condition of lock-in. Stated formally, $T = 0$, for $(\theta_o - \epsilon) < \theta_i < (\theta_o + \epsilon)$.

Examination of Eqs. (17) and (18) indicates that the pull-in time T is

<u>infinite when</u>

$\sec\theta_o = \infty$ or $\theta_o = \dfrac{\pi}{2}$; this corresponds to the case already discussed, in which the detuning $\Delta\omega$ is increased until it equals the open-loop gain K.

$K = 0$; this is a trivial solution corresponding to the open-loop gain $K = 0$.

$\sin\dfrac{\epsilon}{2} = 0$ or $\epsilon = 0$; this has also been discussed previously; it takes infinite time for the loop to pull-in to its steady-state phase error θ_o.

$\cos\dfrac{\theta_i + \theta_o}{2} = 0$, or $\dfrac{\theta_i + \theta_o}{2} = \dfrac{n\pi}{2}$ (n odd). This occurs when $\theta_i = (n\pi - \theta_o)$ and corresponds to the initial phase angle starting at the points of unstable equilibrium given by Eq. (15) for n odd; i.e., $\theta_i = (\pi + \theta_o)$ or $\theta_i = (\pi - \theta_o)$.

<u>zero when</u>

$\theta_i = (\theta_o - \epsilon)$ for Eq. (17) and $\theta_i = (\theta_o + \epsilon)$ for Eq. (18).

The actual definition of pull-in time is somewhat arbitrary, since it depends on a satisfactory choice being made for the value of ϵ. In this report, the time taken for the loop to pull-in to within 5 degrees of its steady state value ($\epsilon = 5$ degrees) will be called the pull-in time.

The expressions for the pull-in time T given by Eqs. (17) and (18) are rather involved; however, they show that T is

inversely proportional to the open-loop gain K. (For the first-order loop discussed in this report. the noise bandwidth B_n is equal to $\frac{K}{2}$ cps, so that the pull-in time is also inversely proportional to the noise bandwidth.)

a nonlinear function of the initial phase angle θ_i

a function of the steady-state phase angle θ_o, which in turn is a function of the ratio:

$$\frac{\text{frequency mistuning}}{\text{open-loop gain}} = \frac{\Delta\omega}{K} \left(\text{i.e., } \sin\theta_o = \frac{\Delta\omega}{K}\right)$$

Equations (17) and (18) can be rearranged so that the product $T \cdot K$ is then a function of the steady-state phase angle θ_o and the initial phase angle θ_i.

Figure 5 shows $T \cdot K$ plotted against the initial phase angle θ_i, for $\theta_o = 0$, 30 and 60 degrees corresponding to

$$\frac{\Delta\omega}{K} = 0, 0.5 \text{ and } 0.866. \tag{19}$$

Figure 5 also shows that certain characteristics regarding the pull-in time T now become apparent.

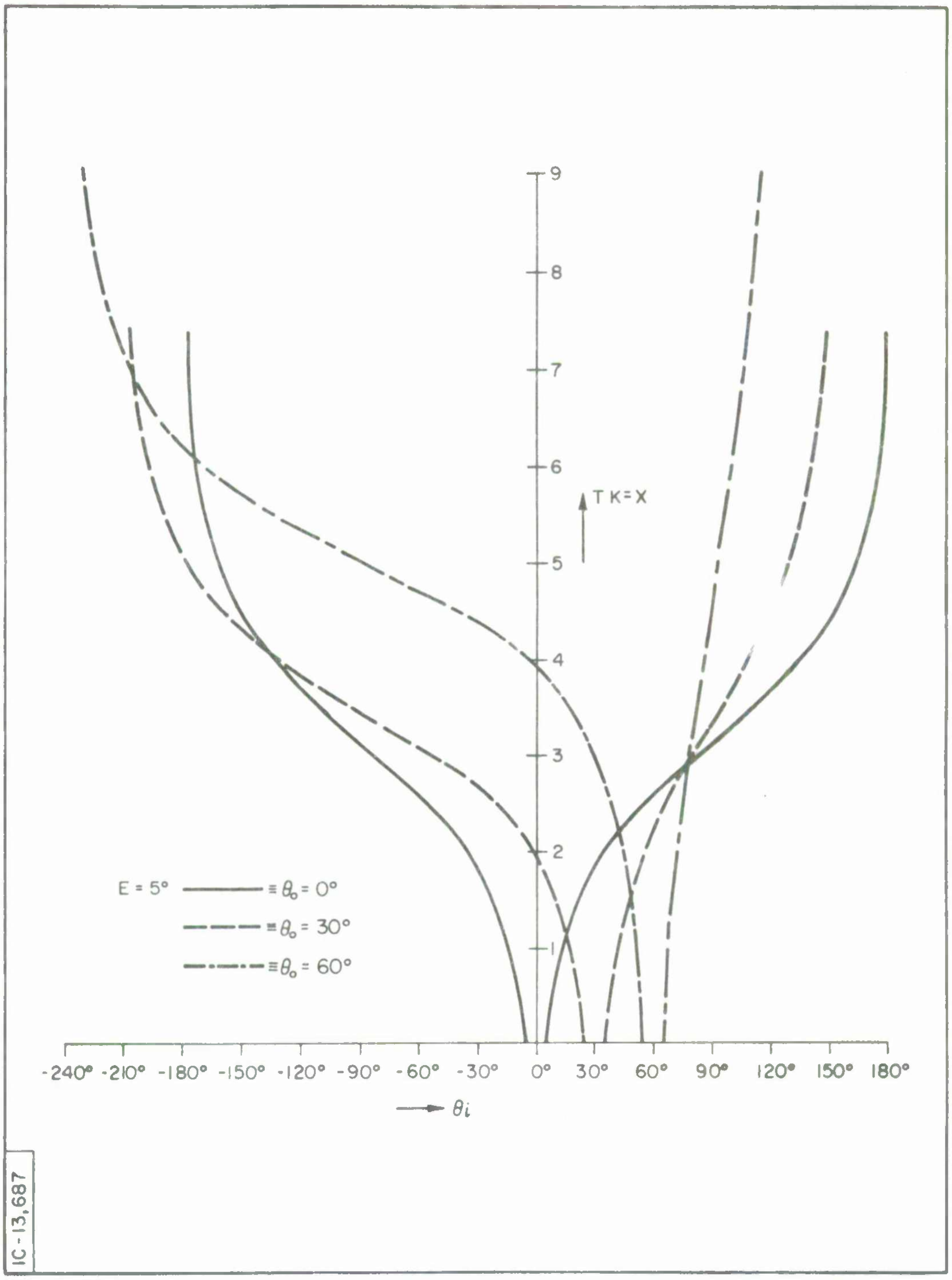

Fig. 5. Plot of Pull-In Time x Open-Loop Gain (T x K) Versus Initial Phase Angle θ_i for $\theta_o = 0°, 30°$ and $60°$

T starts to increase rapidly for initial phase angles approaching $-(\pi + \theta_o)$ and $(\pi - \theta_o)$ for all three values of θ_o. The values $\theta_i = -(\pi + \theta_o)$ and $\theta_i = (\pi - \theta_o)$ are actually asymptotes corresponding to infinite pull-in time.

The two branches of the curve for $\theta_o = 0$ are symmetrical about the line $\theta_i = \theta_o = 0$; as the mistuning increases (θ_o increasing), the two branches become asymmetrical about the line $\theta_i = \theta_o$. There is also a general increase in the pull-in time as θ_o increases.

SECTION V

SUMMARY OF PROBABILITY CONSIDERATIONS ASSOCIATED WITH PULL-IN TIME T FOR FIRST-ORDER PHASE-LOCKED LOOP

The results obtained in Appendixes I, II, III, and IV regarding the statistical nature of the pull-in time are discussed in this section.

In Appendix I (Eqs. (46) and (50)) it is shown that the pull-in time T is given by

$$T = \frac{\sec\theta_o}{K} \ln \left[\frac{\cos\left(\theta_o - \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}} \; \frac{\sin\frac{\theta_o - \theta_i}{2}}{\cos\frac{\theta_o + \theta_i}{2}} \right] \tag{20}$$

for $-(\pi + \theta_o) < \theta_i < (\theta_o - \epsilon)$;

and by

$$T = \frac{\sec\theta_o}{K} \ln \left[\frac{\cos\left(\theta_o + \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}} \; \frac{\sin\left(\frac{\theta_i - \theta_o}{2}\right)}{\cos\left(\frac{\theta_i + \theta_o}{2}\right)} \right] \tag{21}$$

for $\theta_o + \epsilon < \theta_i < (\pi - \theta_o)$.

Let

$$KT = X$$

$$\ln\left[\frac{\cos\left(\theta_o - \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}}\right] = B$$

and

$$\ln\left[\frac{\cos\left(\theta_o + \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}}\right] = B^1$$

Equations (20) and (21) become

$$X = \sec\theta_o \left\{ B + \ln\left[\frac{\sin\frac{\theta_o - \theta_i}{2}}{\cos\frac{\theta_o + \theta_i}{2}}\right]\right\} \qquad (22)$$

for $-(\pi + \theta_o) < \theta_i < (\theta_o - \epsilon)$;

and

$$X = \sec\theta_o \left\{ B^1 + \ln \left[\frac{\sin \dfrac{\theta_i - \theta_o}{2}}{\cos \dfrac{\theta_i + \theta_o}{2}} \right] \right\} \tag{23}$$

for $(\theta_o + \epsilon) < \theta_i < (\pi - \theta_o)$.

The probability density for X (derived in Appendix II) is given by

$$p(X) = \frac{\epsilon}{\pi} \qquad (X = 0) \tag{24}$$

and by

$$p(X) = \frac{\cos^2\theta_o}{2\pi} \left[\frac{1}{\cosh(\cos\theta_o X - B) - \sin\theta_o} + \frac{1}{\cosh(\cos\theta_o X - B - B^1) + \sin\theta_o} \right] \tag{25}$$

$$(X > 0)$$

The probability density, as given by Eq. (25), is shown plotted in Fig. 6 for $\theta_o = 0, 30$ and 60 degrees $\left(\frac{\Delta\omega}{K} = 0, 0.5 \text{ and } 0.866 \right)$. The value of ϵ was again taken as $\epsilon = 5$ degrees.

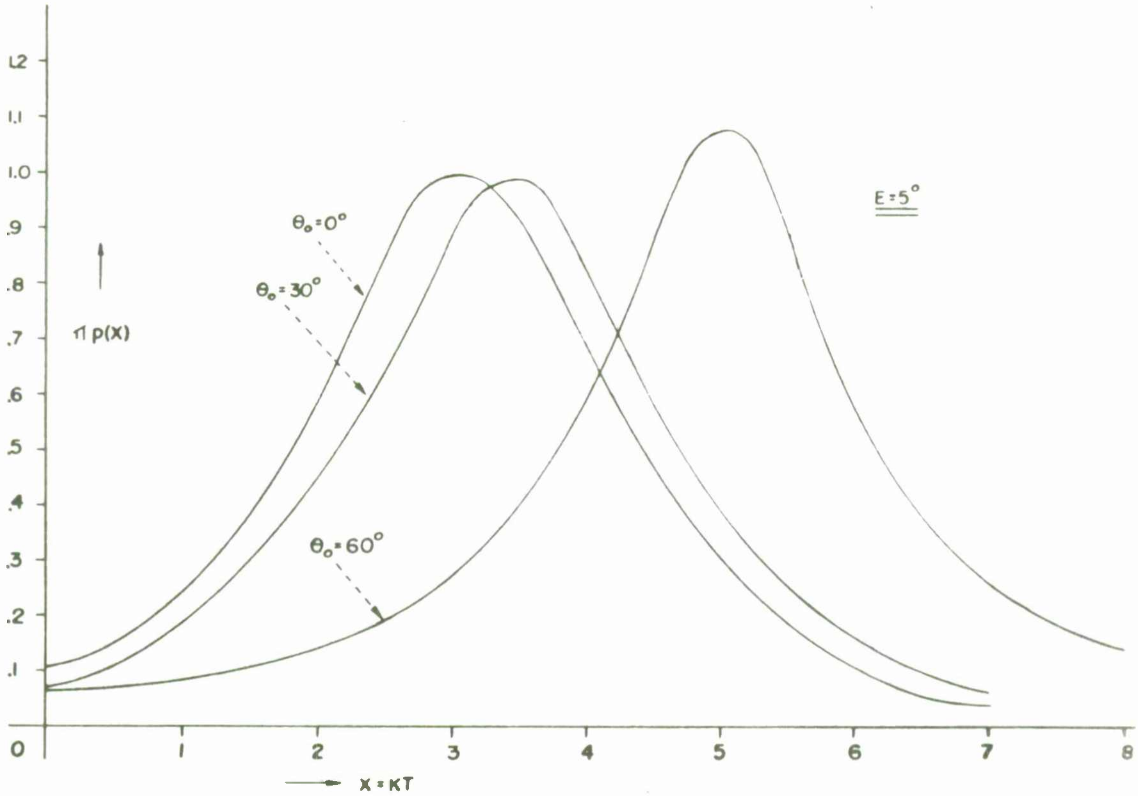

Fig. 6. Probability Density Function for the Product (T x K)

The mean of X, $(\bar{X})$, (Eqs. (98) and (99), Appendix III) is given by

$$\bar{X} = \frac{1}{2\pi} \sec\theta_o \left[B(\pi + 2\theta_o - \epsilon) + B^1 (\pi - 2\theta_o - \epsilon) + 2\int_{\theta_o - \frac{\epsilon}{2}}^{\theta_o + \frac{\epsilon}{2}} \ln(\cos\lambda)d\lambda - 4\int_0^{\frac{\epsilon}{2}} \ln(\sin\lambda)d\lambda \right] \tag{26}$$

If ϵ is small, the last two terms in Eq. (26) may be ignored and the equation reduced to

$$\bar{X} \approx \frac{1}{\pi} \sec\theta_o \left[B(\pi + 2\theta_o) + B^1 (\pi - 2\theta_o)\right] \tag{27}$$

For ϵ small $\quad B \approx B^1 = \ln\left(\cos\theta_o \cot \frac{\epsilon}{2}\right)$, so that

$$\bar{X} \approx \sec\theta_o \ln\left(\frac{2 \cos\theta_o}{\epsilon}\right) \tag{28}$$

for ϵ small.

Figure 7 shows how $\bar{X}$ varies as a function of the steady-state phase angle θ_o, which is related to $\Delta\omega$ and TK by $\frac{\Delta\omega}{K} = \sin^{-1} \theta_o$. $\bar{X}$ has been plotted against θ_o for $\epsilon =$ 5 degrees and 0.5 degrees, and indicates how the mean value of the pull-in time increases as the loop pulls in closer to the steady-state phase angle.

The cumulative distribution for X is derived in Appendix IV and is given by

$$p(X < X_o) = \frac{1}{\pi} \left[\tan^{-1}\left(\frac{e^{\cos\theta_o X_o - B} - \sin\theta_o}{\cos\theta_o}\right) + \tan^{-1}\left(\frac{e^{\cos\theta_o X_o - B^1} + \sin\theta_o}{\cos\theta_o}\right)\right] \tag{29}$$

Figure 8 shows the cumulative distribution of X for $\theta_o =$ 0, 30 and 60 degrees. The value of ϵ is again taken as 5 degrees.

P. Bratt

P. Bratt

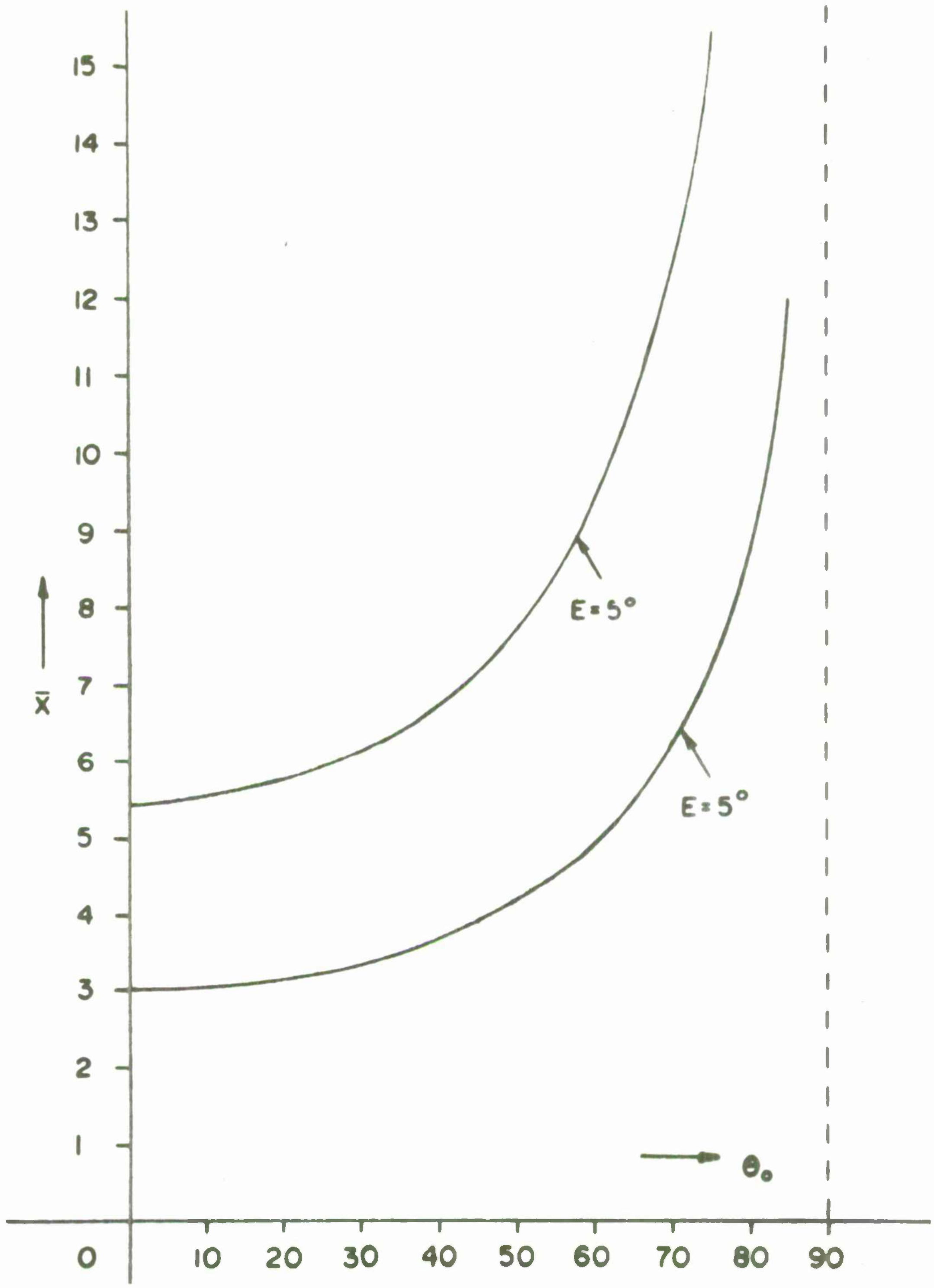

Fig. 7. Plot of $\overline{X}$ - $\overline{K}\overline{T}$ Versus Steady-State Phase Angle

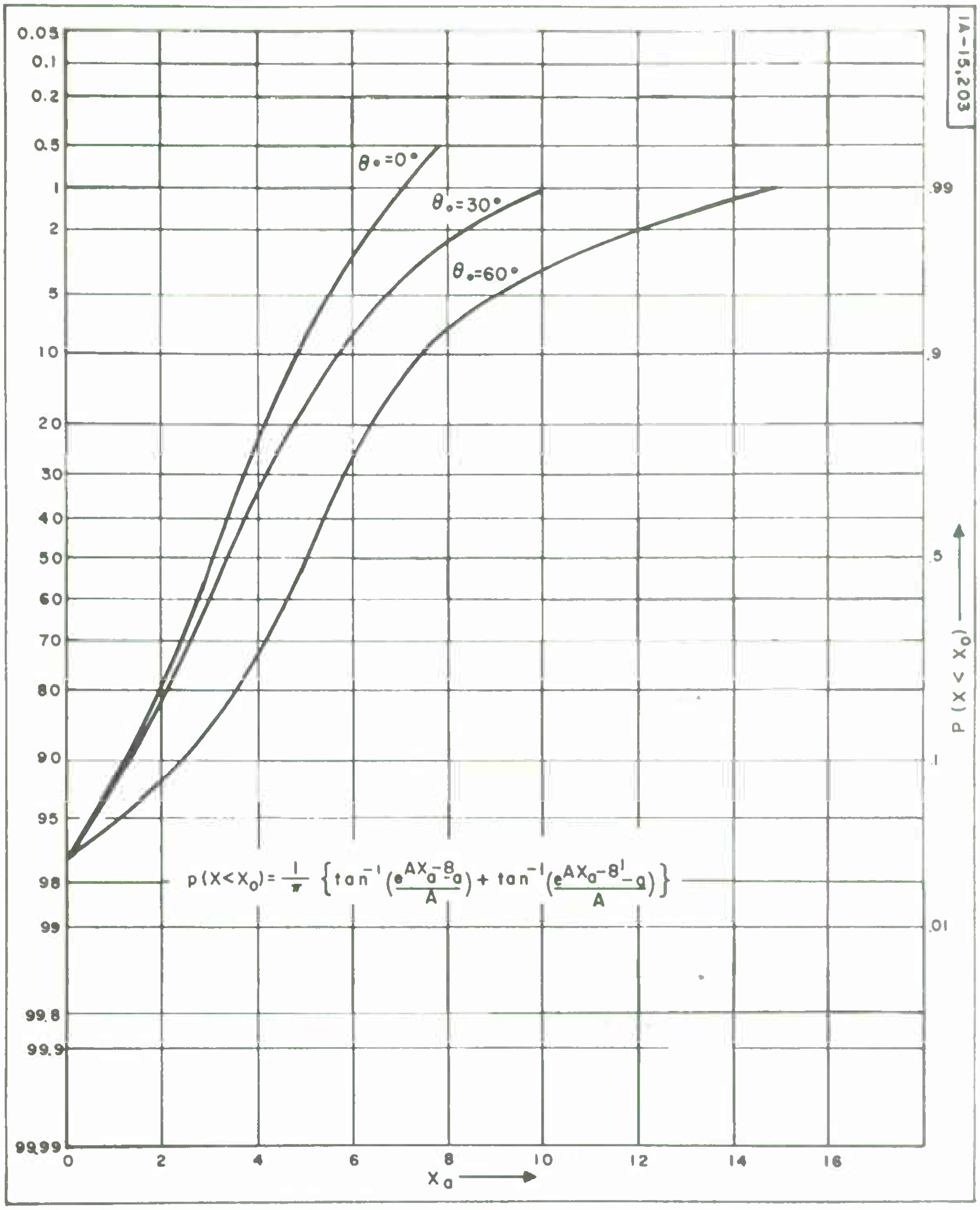

Fig. 8. Cumulative Distribution for X, $\epsilon = 5^{\circ}$

APPENDIX I

DERIVATION OF THE FORMULA RELATING PULL-IN TIME T TO THE INITIAL PHASE ANGLE θ_i

The time required for the phase difference in the loop θ to traverse the phase trajectory shown in Fig. 4 between an initial value θ_i and a final value θ_f is obtained by integrating Eq. (12) and evaluating for θ_i and θ_f. Equation (12) may be rewritten as:

$$\frac{d\theta}{\frac{\Delta\omega}{K} - \sin\theta} = K\,dt$$

or

$$\frac{d\theta}{a - \sin\theta} = K\,dt$$

where

$$a = \frac{\Delta\omega}{K} \tag{30}$$

Integrating Eq. (30) gives

$$\int_{\theta_i}^{\theta_f} \frac{d\theta}{(a - \sin\theta)} = \int_{o}^{T} K\,dt$$

or

$$\frac{1}{\sqrt{1-a^2}} \ln \left[\frac{a \tan \frac{\theta}{2} - \left(1 + \sqrt{1-a^2}\right)}{a \tan \frac{\theta}{2} - \left(1 - \sqrt{1-a^2}\right)} \right] \Bigg|_{\theta_i}^{\theta_f} = KT \tag{31}$$

Let

$$\sqrt{1-a^2} = A$$

$$\therefore \quad \frac{1}{A} \ln \left[\frac{a \tan \frac{\theta}{2} - (1 + A)}{a \tan \frac{\theta}{2} - (1 - A)} \right] \Bigg|_{\theta_i}^{\theta_f} = KT \tag{32}$$

Substituting the limits in Eq. (32) gives

$$\frac{1}{A} \left\{ \ln \left[\frac{a \tan \frac{\theta_f}{2} - (1 + A)}{a \tan \frac{\theta_f}{2} - (1 - A)} - \ln \frac{a \tan \frac{\theta_i}{2} - (1 + A)}{a \tan \frac{\theta_i}{2} - (1 - A)} \right] \right\} = KT \tag{33}$$

Before going further with this equation, consideration should be given to the range of values of θ_i and θ_f for which evaluation of Eq. (33) is of interest.

Basically, we require the time taken for θ to change from an arbitrary initial value θ_i to its final value $\theta_f = \sin^{-1} a = \theta_o$, which is the steady-state phase error, as explained previously. Therefore θ_f in Eq. (33) is simply θ_o, the steady-state phase error.

We now consider the range of initial values of θ_i to be used in evaluating Eq. (33). Figure 4 shows that θ_i falls into two convenient ranges. They are

$$-(\pi + \theta_o) < \theta_i < \theta_o \tag{34}$$

and

$$\theta_o < \theta_i < \pi - \theta_o \tag{35}$$

Returning now to the evaluation of Eq. (33), if the value θ_o is substituted for θ_f in the first term of the left-hand side of the equation the denominator becomes zero and the term itself becomes infinite. This arises because

$$\tan \frac{\theta_o}{2} = \frac{1 - A}{a} \quad (\sin\theta_o = a \text{ and } \cos\theta_o = A)$$

This is to be expected, since as θ approaches θ_o, $\frac{d\theta}{dt}$ tends to zero; e.g., the phase is changing very slowly, requiring infinite time to reach its steady-state value of θ_o. Therefore, the time required for the phase-locked loop to pull into its steady-state value becomes infinite, independent of the initial value of the phase. To avoid this difficulty, the time required for the phase θ to pull-in to within ϵ of the steady-state value θ_o from an arbitrary value of θ_i is determined (see Fig. 4).

Equation (33) is evaluated below in two separate sections according to the two possible ranges of θ_i as given previously, i.e.,

$$-(\pi + \theta_o) \leq \theta_i \leq (\theta_o - \epsilon)$$

$$(\theta_o + \epsilon) \leq \theta_i \leq (\pi - \theta_o)$$

PULL-IN TIME FOR $-(\pi + \theta_o) \leq \theta_i \leq \theta_o - \epsilon$

The first term in Eq. (33) is L where

$$L = \ln \left[\frac{a \tan \frac{\theta_f}{2} - (1 + A)}{a \tan \frac{\theta_f}{2} - (1 - A)} \right] \tag{36}$$

where

$$\theta_f = (\theta_o - \epsilon)$$

Expanding

$$\tan \frac{\theta_f}{2} = \tan \frac{(\theta_o - \epsilon)}{2} = \frac{\tan \frac{\theta_o}{2} - \tan \frac{\epsilon}{2}}{1 + \tan \frac{\theta_o}{2} \cdot \tan \frac{\epsilon}{2}}$$

Now

$$\tan \frac{\theta_o}{2} = \frac{(1 - A)}{a}$$

and let

$$\tan \frac{\epsilon}{2} = \alpha$$

$$\therefore \quad \tan \frac{\theta_f}{2} = \tan \frac{(\theta_o - \epsilon)}{2} = \frac{(1 - A) - a\,\alpha}{a + (1 - A)\alpha} \tag{37}$$

Substituting for $\tan \frac{\theta_f}{2}$ in Eq. (36) gives, after some simplification,

$$L = \ln \left(\frac{a\,(A + a\,\alpha)}{\alpha\,(1 - A)} \right) \tag{38}$$

We now consider the second term in Eq. (33), M, where

$$M = \ln \left[\frac{a \tan \frac{\theta_i}{2} - (1 + A)}{a \tan \frac{\theta_i}{2} - (1 - A)} \right] \tag{39}$$

When considering the initial values of θ, θ_i, it is convenient to use the point of unstable equilibrium at $-(\pi + \theta_o)$ as a reference point in the same way that the final values of θ were measured with reference to the stable point $\theta = \theta_o$.

Referring to Fig. 4 and measuring from the point of unstable equilibrium, it is seen that

$$\theta_i = (\pi + \theta_o) + \delta$$

$$= -\pi - (\theta - \delta)$$

$$\therefore \quad \tan \frac{\theta_i}{2} = \tan \left[\frac{-\pi - (\theta_o - \delta)}{2} \right]$$

$$= \cot \frac{(\theta_o - \delta)}{2} = \frac{1 + \tan \frac{\theta_o}{2} \tan \frac{\delta}{2}}{\tan \frac{\theta_o}{2} - \tan \frac{\delta}{2}}$$

Recalling that $\tan \frac{\theta_o}{2} = \frac{(1 - A)}{a}$, and letting $\tan \frac{\delta}{2} = \beta$,

$$\tan \frac{\theta_i}{2} = \frac{1 + \frac{1 - A}{a} \cdot \beta}{\frac{1 - A}{a} - \beta} = \frac{a + (1 - A)\beta}{(1 - A) - a\beta} \tag{40}$$

Substituting in Eq. (39) for $\tan \frac{\theta_i}{2}$, we obtain, after simplification,

$$M = \ln \left\{ \frac{a\beta}{(1 - A)(A + a\beta)} \right\} \tag{41}$$

Using the values obtained for L and M from Eq. (38) and (41) and substituting in Eq. (33) gives

$$KT = \frac{1}{A}\left\{\ln\left[\frac{a\,(A + a\,\alpha)}{\alpha\,(1 - A)}\right] - \ln\left[\frac{a\,\beta}{(1 - A)\,(A + a\,\beta)}\right]\right\}$$

$$= \frac{1}{A}\ \ln\left[\frac{(A + a\,\alpha)\,(A + a\,\beta)}{\alpha \qquad \beta}\right] \tag{42}$$

$$= \frac{1}{A}\ \ln\left[\left(\frac{A}{\alpha} + a\right)\left(\frac{A}{\beta} + a\right)\right]$$

Recalling that $\alpha = \tan\frac{\epsilon}{2}$ and $\beta = \tan\frac{\delta}{2}$, Eq. (42) becomes

$$KT = \frac{1}{A}\ \ln\left[\left(A\cot\frac{\epsilon}{2} + a\right)\left(A\cot\frac{\delta}{2} + a\right)\right]$$

$$\therefore\ T = \frac{1}{KA}\ \ln\left[\left(A\cot\frac{\epsilon}{2} + a\right)\left(A\cot\frac{\delta}{2} + a\right)\right] \tag{43}$$

Recalling that $A = \cos\theta_o$ and $a = \sin\theta_o$, Eq. (43) may be written as

$$T = \frac{1}{K\cos\theta_o}\ \ln\left[\frac{\cos\left(\theta_o - \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}} \cdot \frac{\cos\left(\theta_o - \frac{\delta}{2}\right)}{\sin\frac{\delta}{2}}\right] \tag{44}$$

Also

$$\theta_i = -(\pi + \theta_o) + \delta$$

for $-(\pi + \theta_o) \leq \theta_i \leq \theta_o - \epsilon$, so that

$$\delta = \theta_i + \pi + \theta_o \tag{45}$$

Equation (44) may now be rewritten by substituting the dependent variable θ_i for δ.

Equation (44) now becomes

$$T = \frac{1}{K \cos \theta_o} \ln \left[\frac{\cos\left(\theta_o - \frac{\epsilon}{2}\right)}{\sin \frac{\epsilon}{2}} \quad \frac{\sin\left(\frac{\theta_o - \theta_i}{2}\right)}{\cos\left(\frac{\theta_o + \theta_i}{2}\right)} \right] \tag{46}$$

PULL-IN TIME FOR $(\theta_o + \epsilon) \leq \theta_i \leq (\pi - \theta_o)$

Using the same approach used in evaluating T for the range $-(\pi + \theta_o) \leq \theta_i \leq (\theta_o - \epsilon)$, but using the point $\theta = (\pi - \theta_o)$ as a reference for the initial value of θ (see Fig. 4), it can be shown that

$$T = \frac{1}{KA} \ln \left[\left(A \cot \frac{\epsilon}{2} - a \right) \left(A \cot \frac{\delta}{2} - a \right) \right] \tag{47}$$

which reduces to

$$T = \frac{1}{K \cos \theta_o} \ln \left[\frac{\cos\left(\theta_o + \frac{\epsilon}{2}\right)}{\sin \frac{\epsilon}{2}} \cdot \frac{\cos\left(\theta_o + \frac{\delta}{2}\right)}{\sin \frac{\delta}{2}} \right] \tag{48}$$

We can rewrite Eq. (48) with θ_i as the dependent variable since

$$\theta_i = \pi - \theta_o - \delta ,$$

for $(\theta_o + \epsilon) \leq \theta_i \leq (\pi - \theta_o)$, or

$$\delta = (\pi - \theta_o - \theta_i) \tag{49}$$

Therefore Eq. (48) becomes

$$T = \frac{1}{K \cos \theta_o} \ln \left[\frac{\cos\left(\theta_o + \frac{\epsilon}{2}\right)}{\sin \frac{\epsilon}{2}} \cdot \frac{\sin\left(\frac{\theta_i - \theta_o}{2}\right)}{\cos\left(\frac{\theta_i + \theta_o}{2}\right)} \right] \tag{50}$$

APPENDIX II

PROBABILITY DENSITY FOR THE PULL-IN TIME FOR A FIRST-ORDER PHASE-LOCKED LOOP

In Appendix I it was shown that the pull-in time for the first-order phase-locked loop is given by (see Eqs. (43) and (47)).

$$KT = \frac{1}{A} \ln \left[\left(A \cot \frac{\epsilon}{2} + a\right) \left(A \cot \frac{\delta}{2} + a\right)\right] \qquad (51)$$

for $-(\pi + \theta_o) \leq \theta_i \leq (\theta_o - \epsilon)$;

and

$$KT = \frac{1}{A} \ln \left[\left(A \cot \frac{\epsilon}{2} - a\right) \left(A \cot \frac{\delta}{2} + a\right)\right] \qquad (52)$$

for $(\theta_o + \epsilon) \leq \theta_i \leq (\pi - \theta_o)$.

Equations (46) and (50) gave the pull-in time with the initial phase angle θ_i as the dependent variable. In this and later calculations on probabilities, it was found that the mathematics was greatly simplified by using δ as the dependent variable. It is then quite simple to substitute for δ in terms of θ_i at the end of the calculations. For convenience we let $KT = X$ in Eqs. (51) and (52), which then become

$$X = \frac{1}{A} \ln \left[\left(A \cot \frac{\epsilon}{2} + a\right) \left(A \cot \frac{\delta}{2} + a\right)\right] \qquad (53)$$

and

$$X = \frac{1}{A} \ln \left[\left(A \cot \frac{\epsilon}{2} - a \right) \left(A \cot \frac{\delta}{2} - a \right) \right] \tag{54}$$

To obtain the probability distribution for X, we assume that all initial phase angles θ_i, $-(\pi + \theta_o) \leq \theta_i \leq (\pi - \theta_o)$ are equally probable; that is, the probability distribution for θ_i is uniform and of value $p(\theta_i) = \frac{1}{2\pi}$ (see Fig. 9).

Since the use of δ as the dependent variable instead of θ_i is simply a linear shift of the angular coordinate, the probability distribution for δ itself is uniform and of value $p(\delta) = \frac{1}{2\pi}$.

The probability distribution $p(X)$ consists of two parts, one discrete and the other continuous. The discrete part consists of an impulse at $X = 0$ magnitude $\frac{2\epsilon}{2\pi} = \frac{\epsilon}{\pi}$. This can be seen by considering Fig. 9, which shows that initial phase angles θ_i such that $\theta_o - \epsilon \leq \theta_i \leq \theta_o + \epsilon$ already lie within the prescribed distance of θ_o and are therefore regarded as synchronous and requiring zero time to pull-in.

The continuous part of the probability distribution is determined below. It is again convenient to break this problem into two sections according to the two possible ranges of θ_i .

$$-(\pi + \theta_o) \leq \theta_i \leq (\theta_o - \epsilon)$$

$$(\theta_o + \epsilon) \leq \theta_i \leq (\pi - \theta_o)$$

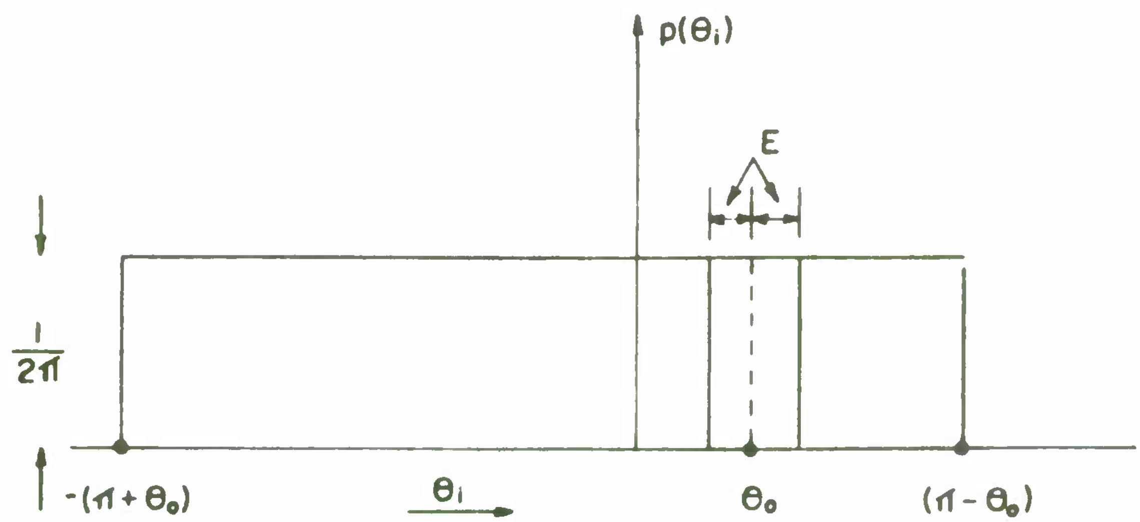

Fig. 9. Probability Distribution for Initial Phase Angle θ_i

PROBABILITY DISTRIBUTION p (X) FOR . $-(\pi + \theta_o) \leq \theta_i \leq (\theta_o - \epsilon)$

Using Eq. (53)

$$X = \frac{1}{A} \ln \left[\left(A \cot \frac{\epsilon}{2} + a\right) \left(A \cot \frac{\delta}{2} + a\right)\right] \tag{55}$$

or

$$X = \frac{B}{A} + \frac{1}{A} \ln \left(A \cot \frac{\delta}{2} + a\right) \tag{56}$$

where

$$B = \ln \left(A \cot \frac{\epsilon}{2} + a \right)$$

is a constant dependent on the mistuning and ϵ, and does not contain the variable δ.

Since X is a continuous single-valued function of θ_i for the range of values of θ_i considered, the probability that the variable X lies between the values x and $x + dx$ is equal to the probability that the variable θ_i lies between the values θ and $\theta + d\theta$. This is also equal to the probability that the variable δ lies between the values θ and $\theta + d\theta$ as previously explained:

$$\begin{aligned} p\,(x \leq X \leq x + dx) &= p\,(\theta \leq \theta_i \leq \theta + d\theta) \\ &= p\,(\theta \leq \delta \leq \theta + d\theta) \end{aligned}$$

In differential notation

$$p\,(X)\; dX = +\, p\,(\delta)\; d\delta$$

or

$$p\,(X) = +\, p\,(\delta) \cdot \frac{1}{\frac{dX}{d\delta}} \tag{57}$$

Since

$$p(\delta) = p(\theta_i) = \frac{1}{2\pi}$$

we require only $\frac{dX}{d\delta}$ in order to evaluate $p \cdot (X)$ using Eq. (57).

Using Eq. (56)

$$X = \frac{B}{A} + \frac{1}{A} \ln \left(A \cot \frac{\delta}{2} + a \right) \tag{58}$$

$$\frac{dX}{d\delta} = \frac{1}{A} \frac{1}{\left(A \cot \frac{\delta}{2} + a \right)} \frac{d}{d\delta} \left(A \cot \frac{\delta}{2} + a \right)$$

$$= -\frac{1}{2} \frac{\operatorname{cosec}^2 \frac{\delta}{2}}{\left(A \cot \frac{\delta}{2} + a \right)} \tag{59}$$

We now require the trigonometric function in Eq. (59) in terms of X . Using Eq. (58)

$$AX - B = \ln \left(A \cot \frac{\delta}{2} + a \right)$$

or

$$
e^{AX-B} = \left(A \cot \frac{\delta}{2} + a\right)
$$

$$
\therefore \quad \frac{e^{AX-B} - a}{A} = \cot \frac{\delta}{2} \tag{60}
$$

Now

$$
\operatorname{cosec}^2 \frac{\delta}{2} = 1 + \cot^2 \frac{\delta}{2}
$$

$$
\therefore \quad \operatorname{cosec}^2 \frac{\delta}{2} = 1 + \left(\frac{e^{AX-B} - a}{A}\right)^2 \tag{61}
$$

Using Eqs. (60) and (61) and substituting in Eq. (59)

$$
\frac{dX}{d\delta} = -\frac{1}{2}\left[\frac{1 + \left(\frac{e^{AX-B} - a}{A}\right)^2}{e^{AX-B}}\right] \tag{62}
$$

Substituting in Eq. (57) for $\frac{dX}{d\delta}$ obtained in Eq. (62) and recalling that $p(\delta) = \frac{1}{2\pi}$

$$
p(X) = \frac{1}{\pi}\left[\frac{e^{AX-B}}{1 + \left(\frac{e^{AX-B} - a}{A}\right)^2}\right] \tag{63}
$$

PROBABILITY DISTRIBUTION $p(X)$ FOR $(\theta_o + \epsilon) \leq \theta_i \leq (\pi - \theta_o)$

Using Eq. (54)

$$X = \frac{1}{A} \ln \left[\left(A \cot \frac{\epsilon}{2} - a \right) \left(A \cot \frac{\delta}{2} - a \right) \right] \tag{64}$$

or

$$X = \frac{B^1}{A} + \frac{1}{A} \ln \left(A \cot \frac{\delta}{2} - a \right) \tag{65}$$

where

$$B^1 = \ln \left(A \cot \frac{\epsilon}{2} - a \right)$$

and is a constant dependent on the mistuning and ϵ, and does not contain the variable δ. The similarity between Eqs. (65) and (66) is apparent, so to obtain $p(X)$ for $\theta_o + \epsilon \leq \theta_i \leq (\pi - \theta_o)$ we just change B to B^1 and $-a$ to $+a$ in Eq. (63) and we have the probability distribution

$$p(X) = \frac{1}{\pi} \left[\frac{e^{AX-B^1}}{1 + \left(\frac{e^{AX-B^1} + a}{A} \right)^2} \right] \tag{66}$$

FINAL EXPRESSION FOR THE PROBABILITY DISTRIBUTION

Using the results obtained from Eqs. (63) and (66) and recalling that

$p(X = 0) = \frac{\epsilon}{\pi}$

$$p(X) = \frac{\epsilon}{\pi} \quad \text{for } X = 0 \tag{67}$$

and

$$p(X) = \frac{1}{\pi}\left[\frac{e^{AX-B}}{1+\left(\frac{e^{AX-B}-a}{A}\right)^2} + \frac{e^{AX-B^1}}{1+\left(\frac{e^{AX-B^1}+a}{A}\right)^2}\right]$$

for $X > 0$ where

$$B = \ln\left(A \cot\frac{\epsilon}{2} + a\right) \tag{69}$$

and

$$B^1 = \ln\left(A \cot\frac{\epsilon}{2} - a\right) \tag{70}$$

The expression for the probability density function given by Eq. (68) may now be rewritten with θ_o as the dependent variable by recalling that

$$a = \sin\theta_o \text{ and } A = \cos\theta_o$$

Equation (68) reduces to

$$p(X) = \frac{\cos^2\theta_o}{2\pi}\left[\frac{1}{\cosh(AX-B) - \sin\theta_o} + \frac{1}{\cosh(AX-B^1) + \sin\theta_o}\right] \tag{71}$$

where

$$B = \ln\left[\frac{\cos\left(\theta_o - \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}}\right] \tag{72}$$

and

$$B^1 = \ln\left[\frac{\cos\left(\theta_o - \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}}\right] \tag{73}$$

Since later appendixes are concerned with the mean and the accumulative distribution, it is necessary to verify that the integral of the probability density does in fact equal unity, that is

$$\int_{0}^{\infty} p\,(X)\,dX = 1$$

Using Eqs. (67) and (68), if $\int_{0}^{\infty} p\,(X) = 1$, it is required to show that

$$\int_{0}^{\infty} \left[\frac{e^{AX-B}}{1 + \left(\frac{e^{AX-B} - a}{A} \right)^2} + \frac{e^{AX-B}}{1 + \left(\frac{e^{AX-B^1} + a}{A} \right)^2} \right] dX = \pi - \epsilon \tag{74}$$

To evaluate the first term in Eq. (74); i.e.,

$$\int_{0}^{\infty} \frac{e^{AX-B}}{1 + \left(\frac{e^{AX-B} - a}{A} \right)^2}\, dX$$

we use the substitution

$$\left(\frac{e^{AX-B} - a}{A} \right)^2 = u$$

Then

$$e^{AX-B}\, dX = du$$

$$\int_0^{\infty} \frac{e^{AX-B}}{1+\left(\dfrac{e^{AX-B}-a}{A}\right)^2}\, dx = \int_{\frac{e^{-B}-a}{A}}^{\infty} \frac{du}{1+u^2}$$

$$= \left|\tan^{-1} u\right|_{\frac{e^{-B}-a}{A}}^{\infty} \tag{75}$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{e^{-B}-a}{A}\right)$$

Similarly the second term in Eq. (74); i.e.,

$$\int_0^{\infty} \frac{e^{AX-B^1}\, dX}{1+\left(\dfrac{e^{AX-B^1}+a}{A}\right)^2} = \frac{\pi}{2} - \tan^{-1}\left(\frac{e^{-B^1}+a}{A}\right) \tag{76}$$

Using Eqs. (75) and (76), the left-hand side of Eq. (74) is equal to

$$\pi - \tan^{-1}\left(\frac{e^{-B}-a}{A}\right) + \tan^{-1}\left(\frac{e^{-B^1}+a}{A}\right)$$

To show that $\int_0^{\infty} p(X) = 1$, it remains to prove that

$$\tan^{-1}\left(\frac{e^{-B} - a}{A}\right) + \tan^{-1}\left(\frac{e^{-B^1} + a}{A}\right) = \epsilon \tag{77}$$

Recalling that

$$e^{-B} = \left(A \cot \frac{\epsilon}{2} + a\right)^{-1} \quad \text{(from Eq. (56))}$$

and

$$e^{-B^1} = \left(A \cot \frac{\epsilon}{2} - a\right)^{-1} \quad \text{(from Eq. (65))}$$

Equation (77) becomes

$$\tan^{-1}\left(\frac{A - a \cot \frac{\epsilon}{2}}{A \cot \frac{\epsilon}{2} + a}\right) + \tan^{-1}\left(\frac{A + a \cot \frac{\epsilon}{2}}{A \cot \frac{\epsilon}{2} - a}\right) = \epsilon \tag{78}$$

For convenience let

$$s = \cot \frac{\epsilon}{2} \tag{79}$$

$$u = \tan^{-1}\left(\frac{A - as}{As + a}\right) \tag{80}$$

$$v = \tan^{-1}\left(\frac{A + as}{As - a}\right) \tag{81}$$

Taking the tangent of each side of Eq. (78) we get

$$\tan(u + v) = \tan \epsilon \tag{82}$$

Now

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \tag{83}$$

Using Eqs. (80) and (81)

$$\begin{aligned}
\tan(u + v) &= \frac{\dfrac{A - as}{As + a} + \dfrac{A + as}{As - a}}{1 - \left(\dfrac{A - as}{As + a}\right)\left(\dfrac{A + as}{As - a}\right)} \\
&= \frac{(A - as)(As - a) + (A + as)(As + a)}{(As + a)(As - a) - (A - as)(A + as)} \\
&= \frac{2s(A^2 + a^2)}{(s^2 - 1)(A^2 + a^2)} \\
&= \frac{2s}{(s^2 - 1)}
\end{aligned} \tag{84}$$

Since $s = \cot \frac{\epsilon}{2}$ (from Eq. (79), we have

$$\tan (u + v) = \frac{2 \cot \frac{\epsilon}{2}}{\cot^2 \frac{\epsilon}{2} - 1} = \frac{2 \tan \frac{\epsilon}{2}}{1 - \tan^2 \frac{\epsilon}{2}}$$

$$\therefore \quad \tan (u + v) = \tan \epsilon$$

which completes the proof required by Eq. (82).

APPENDIX III

DETERMINATION OF MEAN FOR PULL-IN TIME T OF A FIRST-ORDER PHASE-LOCKED LOOP

Appendix II shows that the probability distribution for X is given by

$$p(X) = \frac{\epsilon}{\pi} \quad \text{for } X = 0 \tag{85}$$

and

$$p(X) = \frac{1}{\pi}\left[\frac{e^{AX-B}}{1+\left(\frac{e^{AX-B} - a}{A}\right)^2} + \frac{e^{AX-B^1}}{1+\left(\frac{e^{AX-B^1} + a}{A}\right)^2}\right] \tag{86}$$

for $X > 0$,

where

$X \quad = KT$

$K \quad =$ open-loop gain of the loop (a constant)

$T \quad =$ pull-in time of the loop

The impulse at $X = 0$ does not contribute in any way to the mean; hence it can be ignored. The mean of X is therefore given by

$$\bar{X} = \int_0^\infty X\, p(X)\, dX \tag{87}$$

Using Eq. (86) gives

$$\bar{X} = \frac{1}{\pi} \int_0^\infty \left[\frac{X\, e^{AX-B}}{1 + \left(\frac{e^{AX-B} - a}{A} \right)^2} + \frac{X\, e^{AX-B^1}}{1 + \left(\frac{e^{AX-B^1} + a}{A} \right)^2} \right] dX \tag{88}$$

It is not obvious from Eq. (88) which of the many possible substitutions will give a convenient solution. The trick in obtaining the solution to Eq. (88) is to recall how the expression for $p(X)$ was derived in the first place.

Equations (55) and (64) were used, originally, to determine $p(X)$; i. e.,

$$X = \frac{1}{A} \ln \left[\left(A \cot \frac{\epsilon}{2} + a \right) \left(A \cot \frac{\delta}{2} + a \right) \right] \tag{89}$$

and

$$X = \frac{1}{A} \ln \left[\left(A \cot \frac{\epsilon}{2} - a \right) \left(A \cot \frac{\delta}{2} - a \right) \right] \tag{90}$$

If we substitute X, as given in Eq. (89), in the first term of Eq. (88), and X, as given in Eq. (90), for the second term in Eq. (88), then Eq. (88) reduces to

$$\bar{X} = \frac{1}{2\pi A}\int_{\delta=0}^{\pi - 2\theta_o - \epsilon} \ln\left[\left(A\cot\frac{\epsilon}{2} - a\right)\left(A\cot\frac{\delta}{2} + a\right)\right] d\delta$$

$$+ \int_{\delta=0}^{\pi + 2\theta_o - \epsilon} \ln\left[\left(A\cot\frac{\epsilon}{2} + a\right)\left(A\cot\frac{\delta}{2} + a\right)\right] d\delta \tag{91}$$

where

$$\theta_o = \sin^{-1}(a)$$

Consider the first integral in Eq. (91); i.e.,

$$\int_{\delta=0}^{\pi - 2\theta_o - \epsilon} \ln\left[\left(A\cot\frac{\epsilon}{2} - a\right)\left(A\cot\frac{\delta}{2} - a\right)\right] d\delta$$

$$= \int_{\delta=0}^{\pi - 2\theta_o - \epsilon} \left[\ln\left(A\cot\frac{\epsilon}{2} - a\right) + \ln\left(A\cot\frac{\delta}{2} - a\right)\right] d\delta \tag{92}$$

$$= \ln\left(A\cot\frac{\epsilon}{2} - a\right)(\pi - 2\theta_o - \epsilon) + \int_{\delta=0}^{\pi - 2\theta_o - \epsilon} \ln\left(A\cot\frac{\delta}{2} - a\right) d\delta$$

Consider the second integral in Eq. (91)

$$\int_{\delta=0}^{\pi+2\theta_o-\epsilon} \ln\left[\left(A\cot\frac{\epsilon}{2}+a\right)\left(A\cot\frac{\delta}{2}+a\right)\right]d\delta$$

$$= \int_{\delta=0}^{\pi+2\theta_o-\epsilon}\left[\ln\left(A\cot\frac{\epsilon}{2}+a\right)+\ln\left(A\cot\frac{\delta}{2}+a\right)\right]d\delta \tag{93}$$

$$= \ln\left(A\cot\frac{\epsilon}{2}+a\right)(\pi+2\theta_o-\epsilon)+\int_{\delta=0}^{\pi+2\theta_o-\epsilon}\ln\left(A\cot\frac{\epsilon}{2}+a\right)d\delta$$

Using Eqs. (92) and (93) and substituting in Eq. (91)

$$\begin{aligned}\bar{X} &= \frac{1}{2\pi A}\left[\ln\left(A\cot\frac{\epsilon}{2}-a\right)(\pi-2\theta_o-\epsilon)+\ln\left(A\cot\frac{\epsilon}{2}+a\right)(\pi+2\theta_o-\epsilon)\right]\\ &+\frac{1}{2\pi A}\left[\int_{\delta=0}^{\pi-2\theta_o-\epsilon}\ln\left(A\cot\frac{\delta}{2}-a\right)d\delta+\int_{\delta=0}^{\pi+2\theta_o-\epsilon}\ln\left(A\cot\frac{\delta}{2}+a\right)d\delta\right]\end{aligned} \tag{94}$$

Since $A = \cos\theta_o$, and $a = \sin\theta_o$

$$A\cot\frac{\delta}{2}-a = \frac{\cos\left(\frac{\delta}{2}+\theta_o\right)}{\sin\frac{\delta}{2}} \tag{95}$$

and

$$A \cot \frac{\delta}{2} + a = \frac{\cos\left(\frac{\delta}{2} - \theta_o\right)}{\sin \frac{\delta}{2}} \tag{96}$$

Using these results gives for the second part of Eq. (94)

$$\frac{1}{2 \pi A} \left\{ \int_{\delta = 0}^{\pi + 2\theta_o - \epsilon} \left[\ln \cos\left(\frac{\delta}{2} + \theta_o\right) - \ln \sin \frac{\delta}{2} \right] d\delta \right.$$

$$\left. + \int_{\delta = 0}^{\pi + 2\theta_o - \epsilon} \left[\ln \cos\left(\frac{\delta}{2} - \theta_o\right) - \ln \sin \frac{\delta}{2} \right] d\delta \right\}$$

which, after considerable rearrangement of limits, reduces to

$$\frac{1}{2 \pi A} \quad 2 \int_{\theta_o - \frac{\epsilon}{2}}^{\theta_o + \frac{\epsilon}{2}} \ln \cos (\lambda) \, d\lambda - 4 \int_0^{\frac{\epsilon}{2}} \ln \sin \lambda \, d\lambda \tag{97}$$

Equation (94) now becomes

$$\bar{X} = \frac{1}{2\,\pi\,A} \ln\left(A \cot\frac{\epsilon}{2} - a\right)(\pi - 2\theta_o - \epsilon)$$

$$+ \ln\left(A \cot\frac{\epsilon}{2} + a\right)(\pi + 2\theta_o - \epsilon) \tag{98}$$

$$+ 2\int_{\theta_o - \frac{\epsilon}{2}}^{\theta_o + \frac{\epsilon}{2}} \ln\cos\lambda\, d\lambda - 4\int_0^{\frac{\epsilon}{2}} \ln\sin\lambda\, d\lambda$$

If ϵ is small, the last two terms in Eq. (98) may be neglected and the expression for $\bar{X}$ reduces to

$$\bar{X} \quad \frac{1}{2\,\pi} \sec\theta_o \left\{ \ln\left[\frac{\cos\left(\theta_o + \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}}\right](\pi - 2\theta_o - \epsilon) + \ln\left[\frac{\cos\left(\theta_o - \frac{\epsilon}{2}\right)}{\sin\frac{\epsilon}{2}}\right](\pi + 2\theta_o - \epsilon)\right\} \tag{99}$$

APPENDIX IV

CUMULATIVE DISTRIBUTION FOR THE PULL-IN TIME T

The probability density for the pull-in time T is given by Eqs. (67) and (68); i. e. ,

$$p(X) = \frac{\epsilon}{\pi} \quad \text{for } X = 0 \tag{100}$$

and

$$p(X) = \frac{1}{\pi}\left[\frac{e^{AX-B}}{1+\left(\frac{e^{AX-B}-a}{A}\right)^2} + \frac{e^{AX-B^1}}{1+\left(\frac{e^{AX-B^1}+a}{A}\right)^2}\right] \tag{101}$$

for $X > 0$,

where

$$X = KT$$

and

$$K = \text{open-loop gain of the loop.}$$

The cumulative distribution for X is given by the probability that X is less than some specified value, say X_o; i.e.,

$$p\,(X < X_o) \int_0^{X_o} p\,(X)\, dX$$

Using Eqs. (100) and (101)

$$p\,(X < X_o) = \frac{1}{\pi}\left\{\epsilon + \int_0^{X_o}\left[\frac{e^{AX-B}}{1+\left(\dfrac{e^{AX-B}-a}{A}\right)^2} + \frac{e^{AX-B^1}}{1+\left(\dfrac{e^{AX-B^1}+a}{A}\right)^2}\right] dX\right\} \tag{102}$$

Consider the first integral in Eq. (102)

$$I_1 = \int_0^{X_o} \frac{e^{AX-B}}{1+\left(\dfrac{e^{AX-B}-a}{A}\right)^2}\, dX$$

Let

$$\frac{e^{AX-B}-a}{A} = u$$

$$I_1 = \int_{u_1}^{u_2} \frac{du}{1+u^2}$$

where

$$u_1 = \left(\frac{e^{-B} - a}{A}\right)$$

and

$$u_2 = \left(\frac{e^{AX_o - B} - a}{A}\right)$$

$$I_1 = \left|\tan^{-1} u\right|_{u_1}^{u_2} = \tan^{-1}\left(\frac{e^{AX_o-B} - a}{A}\right) - \tan^{-1}\left(\frac{e^{-B} - a}{A}\right) \qquad (103)$$

Using a similar substitution the second integral in Eq. (102) reduces to

$$I_2 = \tan^{-1}\left(\frac{e^{AX_o-B^1} + a}{A}\right) - \tan^{-1}\left(\frac{e^{-B^1} + a}{A}\right) \qquad (104)$$

Using the results of Eqs. (103) and (104) we obtain

$$p\,(X < X_o) = \frac{1}{\pi}\left[\epsilon + \tan^{-1}\left(\frac{e^{AX_o-B} - a}{A}\right) + \tan^{-1}\left(\frac{e^{AX_o-B^1} + a}{A}\right)\right.$$
$$\left. - \tan^{-1}\left(\frac{e^{-B} - a}{A}\right) - \tan^{-1}\left(\frac{e^{-B^1} + a}{A}\right)\right] \qquad (105)$$

In Appendix II (Eq. (77) onward), it is shown that the last two terms in Eq. (105) were equal to ϵ, so that Eq. (105) reduces to

$$p\,(X < X_o) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{e^{AX_o - B} - a}{A}\right) + \tan^{-1}\left(\frac{e^{AX_o - B^1} + a}{A}\right)\right] \qquad (106)$$

APPENDIX V

A NOTE ON INSTANTANEOUS PHASE AND INSTANTANEOUS FREQUENCY

This Appendix explains the terms "instantaneous phase difference" and "instantaneous frequency difference."

Suppose we are given a sinusoidal waveform

$$E_1 = \sin\theta_1 (t) \tag{107}$$

We define its instantaneous phase as $\theta_1 (t)$ and its instantaneous frequency as the time derivative of its phase angle

$$\dot{\theta}_1 (t) = \frac{d\theta_1 (t)}{dt} \tag{108}$$

For example, consider the case where

$$\theta_1 (t) = \omega_1 t + \alpha \tag{109}$$

which corresponds to a sinusoid with constant angular frequency and an initial starting phase at time $t = 0$ of α.

Equation (109) gives its instantaneous phase angle as a function of time and the time derivative of Eq. (109) gives its instantaneous frequency as a function of time.

$$\dot{\theta}_1 (t) = \omega_1 \tag{110}$$

For this case the word instantaneous is redundant, since we are considering a sine wave of constant frequency.

Suppose now we are given two sinusoidal waveforms

$$E_1 = \sin\theta_1(t) \tag{111}$$

$$E_2 = \sin\theta_2(t) \tag{112}$$

We define their instantaneous phase difference as

$$\theta(t) = \theta_1(t) - \theta_2(t) \tag{113}$$

and their instantaneous frequency difference as

$$\dot{\theta}(t) = \dot{\theta}_1(t) - \dot{\theta}_2(t) \tag{114}$$

For example, consider the case where

$$\theta_1(t) = \omega_1 t + \alpha$$

and

$$\theta_2(t) = \omega_2 t + \beta$$

corresponding to two sinusoids with different angular frequencies and different initial phase angles.

Using Eq. (113), their instantaneous phase difference is

$$\begin{aligned} \theta(t) &= \theta_1(t) - \theta_2(t) \\ &= (\omega_1 t + \alpha) - (\omega_2 t + \beta) \\ &= (\omega_1 - \omega_2)t + (\alpha - \beta) \end{aligned} \tag{115}$$

Using Eq. (114), their instantaneous frequency difference is

$$\begin{aligned} \dot{\theta}(t) &= \dot{\theta}_1(t) - \dot{\theta}_2(t) \\ &= (\omega_1 - \omega_2) \end{aligned} \tag{116}$$

The definitions of instantaneous phase difference and instantaneous frequency difference given by Eqs. (113) and (114) are always valid and independent of the actual form of $\theta_1(t)$ or $\theta_2(t)$.

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13. ABSTRACT

A phase-locked loop is briefly described, together with the derivation of the basic differential equation which governs the dynamic behavior of the loop during the pull-in process. The special case of the pull-in process of the first-order loop when a sine wave of constant frequency is applied to the input of the loop is also described. The relationship between the frequency mistuning of the loop, the initial starting phase angle of the input sine wave, and the time required for the loop to pull in is discussed. The statistical parameters associated with the pull-in time is reviewed. In particular, expressions are given for the probability density function of the pull-in time and the cumulative distribution of the pull-in time.

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ERRATA

ESD-TDR-64-640 Pull-In Performance of First-Order Phase-Locked Loops

Electronic Systems Division

List of Symbols

A	rms amplitude of the input sine wave
A	cos θ_o
a	sine θ_o
B	a constant relating θ_o and ϵ
B^1	a constant relating θ_o and ϵ
C	rms amplitude of the output of the voltage-controlled oscillator
Υ	an angle relating θ_o and θ_i
$\Delta\omega$	$(\omega_1 - \omega_o)$
ϵ	a small angle
θ	instantaneous phase difference
$\dot{\theta}$	instantaneous angular frequency difference
θ_f	final phase angle
θ_i	initial phase angle
θ_o	steady-state phase angle
$\theta(t)$	phase angle as a function of time
K	loop gain

K_1	amplifier gain
K_θ	phase detector gain
K_{vco}	voltage-controlled oscillator gain
ln	$\log_e$
λ	dummy variable
T	pull-in time
X	KT=product of loop gain and pull-in time
$\overline{X}$	average value of X
X_o	specific value of X
ω_o	free-running angular frequency of the voltage-controlled oscillator
ω_1	angular frequency of the input to the phase-locked loop